

# HOMOTOPY TRANSFER AND RATIONAL MODELS FOR MAPPING SPACES

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**ABSTRACT.** We prove that there is an explicit  $L_\infty$ -structure on the complex of linear maps  $\text{Hom}(H, L)$ , where  $H$  is the homology of a cocommutative coalgebra with the transferred  $A_\infty$ -coalgebra structure of higher order Massey coproducts and  $L$  is an  $L_\infty$ -algebra, providing an  $L_\infty$ -model for the rational homotopy type of mapping spaces. By using these homotopy transfer techniques, we give bounds for the nilpotency order of mapping spaces, and conditions on the source and target to detect rational  $H$ -space structures on the components.

## 1. INTRODUCTION

The version up to homotopy of differential graded Lie algebras, called  $L_\infty$ -algebras, was first introduced in the context of deformation theory [30] and highly used since then in different geometrical settings [10, 22]. Recently,  $L_\infty$ -algebras have become a useful tool to describe rational homotopy types of spaces [9, 16]. In particular, mapping spaces seem to be well described in this terms [2, 4, 8, 25].

More concretely, if  $C$  is a coalgebra model of a finite nilpotent CW-complex  $X$  and  $L$  is an  $L_\infty$ -model of a finite type rational CW-complex  $Y$ , then the complex of linear maps  $\text{Hom}(C, L)$  admits an  $L_\infty$ -structure whose geometrical realization is of the rational homotopy type of  $\text{map}(X, Y)$ , the space of continuous functions; see [2, 8] for details. Similarly, the complex  $\text{Hom}(\overline{C}, L)$ , where  $\overline{C}$  is the kernel of the augmentation  $\varepsilon: C \rightarrow \mathbb{Q}$  is an  $L_\infty$ -model for  $\text{map}^*(X, Y)$ , the space of pointed continuous functions.

The aim of this paper is to describe an explicit  $L_\infty$ -structure in  $\text{Hom}(H, L)$ , where  $H = H_*(C)$  is the homology of a coalgebra  $C$ , that serves as a model for mapping spaces. The strategy will be to use the homotopy retract between  $C$  and  $H_*(C)$ , defining the higher Massey coproducts, and the above explicit  $L_\infty$ -structure on  $\text{Hom}(C, L)$  to induce another homotopy retract between  $\text{Hom}(C, L)$  and  $\text{Hom}(H, L)$ . Then, we apply the Homotopy Transfer Theorem [15, 23, 24, 26, 28] to give an  $L_\infty$ -structure on  $\text{Hom}(H, L)$  and an explicit formula for it by means of rooted trees (see Section 3 for further details).

We prove that, indeed, this  $L_\infty$ -structure exhibits  $\text{Hom}(H, L)$  as an  $L_\infty$ -model for  $\text{map}(X, Y)$ . Note, however, that a quasi-isomorphism between two  $L_\infty$ -algebras

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is not necessarily a quasi-isomorphism after applying the generalized cochain functor  $\mathcal{C}^\infty$ . Therefore, we cannot deduce directly from the quasi-isomorphism provided by the homotopy retract that  $\text{Hom}(H, L)$  is an  $L_\infty$ -model for the mapping space. The same argument can be applied by replacing  $(C, \Delta)$  and  $H$  with  $(\overline{C}, \overline{\Delta})$  and  $\overline{H}$ , respectively, to model the space of pointed continuous functions  $\text{map}^*(X, Y)$ .

We provide two different applications. The first one gives a bound for the rational nilpotency order of the mapping space  $\text{map}(X, Y)$  when  $X$  is a finite nilpotent CW-complex and  $Y$  is a finite type rational CW-complex. More precisely, we prove that if  $f: X \rightarrow Y$  denotes any map, then

$$\max\{\text{nil}(\text{map}_f^*(X, Y)), \text{nil}(\text{map}_f(X, Y))\} \leq \text{nil}(Y),$$

where  $\text{nil}(-)$  stands for the rational nilpotency order and  $\text{map}_f(X, Y)$  denotes the component containing the map  $f$ . This improves [8, Corollary 4.3], where the extra conditions of  $X$  being a formal space and  $f$  being the constant map were required.

Finally, we give a necessary condition for the components of mapping spaces to be of the rational homotopy type of an  $H$ -space. This problem has been previously considered in [4, 6, 14]. We prove a variant of the results obtained in these papers in terms of the cone length ( $\text{cl}$ ), the Whitehead length ( $\text{Wl}$ ), the bracket length ( $\text{bl}$ ) and the differential length ( $\text{dl}_k$ ), that does not implicitly assume the coformality of the target space. Explicitly, we prove that if  $\text{cl}(X) < \text{dl}_3(Y)$  and  $\text{Wl}(Y) < \text{bl}(X)$ , then all the components of  $\text{map}^*(X, Y)$  are rationally  $H$ -spaces.

## 2. PRELIMINARIES

We will rely on known results from rational homotopy theory for which [13] is a standard reference. We also assume the reader is aware of the concepts of homotopy operadic algebras being [26] an excellent reference. With the aim of fix notation we give some definitions and sketch some results we will need. Every algebraic object considered throughout the paper is assumed to be a graded vector space over the rationals.

**2.1.  $A_\infty$ -coalgebras and  $L_\infty$ -algebras.** An  $A_\infty$ -coalgebra  $C$  is a graded vector space together with a differential graded algebra structure on the tensor algebra  $T^+(s^{-1}C)$  on the desuspension of  $C$ . This is equivalent to the existence of a family of degree  $k - 2$  linear maps  $\Delta_k: C \rightarrow C^{\otimes k}$  satisfying the equation

$$\sum_{k=1}^i \sum_{n=0}^{i-k} (-1)^{k+n+kn} (\text{id}^{\otimes i-k-n} \otimes \Delta_k \otimes \text{id}^{\otimes n}) \Delta_{i-k+1} = 0.$$

Any differential graded coalgebra  $(C, \delta, \Delta)$  is an  $A_\infty$ -coalgebra with  $\Delta_1 = \delta$ ,  $\Delta_2 = \Delta$  and  $\Delta_k = 0$  for  $k > 2$ . We will denote by  $\Delta^{(k)} = (\Delta \otimes \text{id} \otimes \cdots \otimes \text{id}) \circ \cdots \circ (\Delta \otimes \text{id}) \circ \Delta$  with  $k$  factors, and  $\Delta^{(0)} = \text{id}$ .

An  $A_\infty$ -coalgebra is *cocommutative* if  $\tau \circ \Delta_k = 0$  for every  $k \geq 1$ , where  $\tau: T(C) \rightarrow T(C) \otimes T(C)$  denotes the *unshuffle coproduct*, that is,

$$\tau(a_1 \otimes \cdots \otimes a_n) = \sum_{i=1}^n \sum_{\sigma \in S(i, n-i)} \epsilon_\sigma (a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(i)}) \otimes (a_{\sigma(i)} \otimes \cdots \otimes a_{\sigma(n)}),$$

where  $\epsilon_\sigma$  is the signature of  $\sigma$  and  $S(i, n-i)$  denotes the set of  $(i, n-i)$ -shuffles, i.e., permutations  $\sigma$  of  $n$ -elements such that  $\sigma(1) < \cdots < \sigma(i)$  and  $\sigma(i+1) < \cdots < \sigma(n)$ .

Let  $(C, \{\Delta_k\})$  and  $(C', \{\Delta'_k\})$  be two  $A_\infty$ -coalgebras. A *morphism of  $A_\infty$ -coalgebras* from  $C$  to  $C'$  is a morphism  $f: C \rightarrow C'$  compatible with  $\Delta_k$  and  $\Delta'_k$ . An  *$A_\infty$ -morphism* from  $C$  to  $C'$  is a morphism  $f: T^+(s^{-1}C) \rightarrow T^+(s^{-1}C')$  of differential graded algebras. This is equivalent to the existence of a family of degree  $k-1$  maps  $f^{(k)}: C \rightarrow C'^{\otimes k}$  satisfying the usual relations involving  $\Delta_k$  and  $\Delta'_k$ . An  $A_\infty$ -morphism is a *quasi-isomorphism* if  $f^{(1)}$  is a quasi-isomorphism of complexes.

An  $L_\infty$ -algebra or *strongly homotopy Lie algebra* is a graded vector space  $L$  together with a differential graded coalgebra structure on  $\Lambda^+sL$ , the cofree graded cocommutative coalgebra generated by the suspension. The existence of this structure on  $\Lambda^+sL$  is equivalent to the existence of degree  $k-2$  linear maps  $\ell_k: L^{\otimes k} \rightarrow L$ , for  $k \geq 1$ , satisfying the following two conditions:

- (i) For any permutation  $\sigma$  of  $k$  elements,

$$\ell_k(x_{\sigma(1)}, \dots, x_{\sigma(k)}) = \epsilon_\sigma \ell_k(x_1, \dots, x_k),$$

where  $\epsilon_\sigma$  is the signature of the permutation and  $\epsilon$  is the sign given by the Koszul convention.

- (ii) The *generalized Jacobi identity* holds, that is

$$\sum_{i+j=n+1} \sum_{\sigma \in S(i, n-i)} \epsilon_\sigma \epsilon (-1)^{i(j-1)} \ell_{n-i}(\ell_i(x_{\sigma(1)}, \dots, x_{\sigma(i)}), x_{\sigma(i+1)}, \dots, x_{\sigma(n)}) = 0,$$

where  $S(i, n-i)$  denotes the set of  $(i, n-i)$ -shuffles.

Every differential graded Lie algebra  $(L, \partial)$  is an  $L_\infty$ -algebra by setting  $\ell_1 = \partial$ ,  $\ell_2 = [-, -]$  and  $\ell_k = 0$  for  $k > 2$ . An  $L_\infty$ -algebra  $(L, \{\ell_k\})$  is called *minimal* if  $\ell_1 = 0$ .

Let  $(L, \{\ell_k\})$  and  $(L', \{\ell'_k\})$  be two  $L_\infty$ -algebras. A *morphism of  $L_\infty$ -algebras* from  $L$  to  $L'$  is a morphism  $f: L \rightarrow L'$  that is compatible with  $\ell_k$  and  $\ell'_k$ . An  *$L_\infty$ -morphism* from  $L$  to  $L'$  is a morphism  $f: \Lambda^+sL \rightarrow \Lambda^+sL'$  of differential graded coalgebras.

Any  $L_\infty$ -morphism is completely determined by the projection  $\pi f: \Lambda^+sL \rightarrow sL'$  which is the sum of a system of skew-symmetric morphisms  $f^{(k)}: L^{\otimes k} \rightarrow L'$  of degree  $1-k$ . The morphisms  $f^{(k)}$  satisfy an infinite sequence of equations involving the brackets  $\ell_k$  and  $\ell'_k$  (see for example [22]). In particular, for  $k=1$ , we have that  $\ell'_1 f^{(1)} - f^{(1)} \ell_1 = 0$ . Therefore  $f^{(1)}: (L, \ell_1) \rightarrow (L', \ell'_1)$  is a map of complexes. Any morphism of  $L_\infty$ -algebras is an  $L_\infty$ -morphism of  $L_\infty$ -algebras. As in the case of  $A_\infty$ -coalgebras, an  $L_\infty$ -morphism is a *quasi-isomorphism* if  $f^{(1)}$  is a quasi-isomorphism of complexes. The following result can be found in [22, Theorem 4.6].

**Theorem 2.1.** *Let  $L$  and  $L'$  be two  $L_\infty$ -algebras. If  $f$  is a quasi-isomorphism of  $L_\infty$ -algebras from  $L$  to  $L'$ , then there exists another  $L_\infty$ -morphism from  $L'$  to  $L$  inducing the inverse isomorphism between homology of complexes  $(L, \ell_1)$  and  $(L', \ell'_1)$ .*

An  $L_\infty$ -algebra  $(L, \{\ell_k\})$  is called *linear contractible* if  $\ell_k = 0$  for  $k \geq 2$  and  $H_*(L, \ell_1) = 0$ . Since the property of being linear contractible is not invariant under  $L_\infty$ -isomorphisms, we say that  $L$  is *contractible* if  $L$  is isomorphic as an  $L_\infty$ -algebra to a linear contractible one. As stated in [22, Lemma 4.9], any  $L_\infty$ -algebra is  $L_\infty$ -isomorphic to the direct sum of a minimal  $L_\infty$ -algebra and a linear contractible one. Following [22], one can prove that any quasi-isomorphism between minimal  $L_\infty$ -algebras is an isomorphism. Thus, the set of equivalence classes of  $L_\infty$ -algebras

up to quasi-isomorphisms can be identified with the set of equivalence classes of minimal  $L_\infty$ -algebras up to  $L_\infty$ -isomorphisms.

The *Maurer–Cartan set* of an  $L_\infty$ -algebra  $L$  is the set of elements  $z \in L_{-1}$  such that the infinite series

$$\sum_{k \geq 1} \frac{1}{k!} \ell_k(z, \dots, z)$$

is a finite sum and it is equal to zero. We will denote the set of Maurer–Cartan elements in  $L$  by  $\text{MC}(L)$ . If  $L$  is an  $L_\infty$ -algebra and  $z \in \text{MC}(L)$ , then

$$\ell_k^z(x_1, \dots, x_k) = \sum_{i \geq 0} \frac{1}{i!} \ell_{i+k}(z, \dots, z, x_1, \dots, x_k)$$

defines a new  $L_\infty$ -structure denoted by  $L^z$  whenever the series is a finite sum (cf. [16, Proposition 4.4]). The *perturbed and truncated*  $L_\infty$ -structure on  $L$  is the  $L_\infty$ -algebra  $L^{(z)}$ , whose underlying graded vector space is given by

$$\left(L^{(z)}\right)_i = \begin{cases} L_i & \text{if } i > 0, \\ Z_{\ell_1^z}(L_0) & \text{if } i = 0, \\ 0 & \text{if } i < 0, \end{cases}$$

where  $Z_{\ell_1^z}(L_0)$  denotes the space of cycles for the differential  $\ell_1^z$ , and with the same brackets as  $L^z$ .

If  $L$  is a finite type graded vector space, then an  $L_\infty$ -algebra structure on  $L$  is the same as a commutative differential graded algebra structure on  $\Lambda(sL)^\sharp$ , denoted by  $\mathcal{C}^\infty(L)$ , where  $\sharp$  stands for the dual vector space. More explicitly, if  $V$  and  $sL$  are dual graded vector spaces, then  $\mathcal{C}^\infty(L) = (\Lambda V, d)$  with  $d = \sum_{j \geq 1} d_j$  and

$$(2.1) \quad \langle d_j v; s x_1 \wedge \dots \wedge s x_j \rangle = (-1)^\epsilon \langle v; s \ell_j(x_1, \dots, x_j) \rangle,$$

where  $\langle -; - \rangle$  is defined as an extension of the pairing induced by the isomorphism between  $V$  and  $(sL)^\sharp$ ,  $d_j v \in \Lambda^j V$  and  $\epsilon$  is the sign given by the Koszul convention.

Conversely, if  $(\Lambda V, d)$  is a commutative differential graded algebra of finite type, then an  $L_\infty$ -algebra structure on  $s^{-1}V^\sharp$  is uniquely determined by the condition  $(\Lambda V, d) = \mathcal{C}^\infty(L)$ . Moreover, in the above correspondence,  $(\Lambda V, d)$  is a Sullivan algebra if and only if  $L$  is concentrated in non negative degrees and  $L_0$  acts nilpotently on  $L$  via  $\ell_2$ .

Note that the definition of minimal  $L_\infty$ -algebra is compatible with the one coming from the minimality of Sullivan algebras when  $L$  is of finite type, non negatively graded and with a nilpotent action of  $L_0$ . In this case,  $L$  is minimal if and only if  $\mathcal{C}^\infty(L)$  is a minimal Sullivan algebra. From now on we will always assume that  $L_\infty$ -algebras are of *finite type*, since all the  $L_\infty$ -algebras appearing in the main results of subsequent sections will satisfy this assumption.

If  $L$  is an  $L_\infty$ -algebra, then the *lower central series*  $\{F^k L\}$  of  $L$  is defined inductively by setting  $F^1 L = L$  and

$$F^k L = \sum_{k_1 + \dots + k_i = k} \ell_k(F^{k_1} L, \dots, F^{k_i} L)$$

for  $k \geq 2$ . We say that  $L$  is *nilpotent* if  $F^k L = 0$  for some  $k \geq 1$ , and  $L$  is of *nilpotency order*  $i_0$  if  $F^k L = 0$  for  $k > i_0$  and  $F^{i_0} L \neq 0$ . We will denote by  $\text{nil}(L)$  the nilpotency order of  $L$ . If  $L$  is not nilpotent, we will write  $\text{nil}(L) = \infty$ . The nilpotency order is not an invariant, since isomorphic  $L_\infty$ -algebras can have different nilpotency orders, but if  $L$  and  $L'$  are isomorphic nilpotent *minimal*  $L_\infty$ -algebras of

finite type and concentrated in non negative degrees, then  $\text{nil}(L) = \text{nil}(L')$ . Indeed, higher brackets  $\ell_k$  are identified with  $d_k$  by (2.1) and the latter represent higher order Whitehead products [1].

**2.2. Rational models for mapping spaces.** In [31] Sullivan associated to each nilpotent space  $Z$  a commutative differential graded algebra  $A_{PL}(Z)$ . In fact, there is an adjoint pair

$$A_{PL}: \mathbf{sSets}^{\text{op}} \rightleftarrows \mathbf{CDGA}: \langle - \rangle ,$$

where  $\mathbf{CDGA}$  is the category of commutative differential graded algebras,  $\mathbf{sSets}$  is the category of simplicial sets, and  $\langle - \rangle$  denotes the simplicial realization. The *minimal model* of  $Z$  is defined as the minimal model  $(\Lambda V, d)$  of  $A_{PL}(Z)$ . A *model* of  $Z$  is a graded commutative differential algebra quasi-isomorphic to its minimal model. For more details, we refer to [13, 31].

By a model of a not necessarily connected space  $Z$ , such that all its components are nilpotent (or a map between them), we mean a  $\mathbb{Z}$ -graded commutative differential graded algebra (or a morphism) whose simplicial realization, in the sense of [31], has the same homotopy type as the singular simplicial approximation of  $Z_{\mathbb{Q}}$ . Similarly, by an  $L_{\infty}$ -model of a space  $Z$  as above, we mean an  $L_{\infty}$ -algebra  $L$  such that  $C^{\infty}(L)$  is a CDGA model of  $Z$ .

**Proposition 2.2** ([2, 3, 7]). *Let  $L$  be an  $L_{\infty}$ -algebra and  $z \in L_{-1}$  a Maurer–Cartan element. Then there is a homotopy equivalence*

$$\langle C^{\infty}(L^{(z)}) \rangle \xrightarrow{\simeq} \langle C^{\infty}(L) \rangle_z ,$$

where  $\langle C^{\infty}(L) \rangle_z$  is the connected component containing the 0-simplex associated to  $z$ .

Note that this generalizes the notion of differential graded Lie model of a finite type nilpotent space  $Z$ , since in this case  $C^{\infty}(L) = C^*(L)$ , where  $C^*$  is the classical cochain functor and the only Maurer–Cartan element in  $L$  is the zero element.

Similarly, we say that an  $A_{\infty}$ -coalgebra model of  $Z$  is a cocommutative  $A_{\infty}$ -coalgebra  $C$  such that  $\mathcal{L}_{\infty}(C)$  is a differential graded Lie model of  $Z$ , where  $\mathcal{L}_{\infty}$  denotes the *generalized Quillen functor* (see [7] for further details). This functor assigns to a cocommutative  $A_{\infty}$ -coalgebra  $C$  an induced differential graded Lie algebra structure on  $\mathbb{L}(s^{-1}C)$  whose differential  $\partial = \sum_{k \geq 1} \partial_k$  with  $\partial_k: s^{-1}C \rightarrow \mathbb{L}^k(s^{-1}C)$  is determined by  $\Delta_k$  in the same way as the classical Quillen functor  $\mathcal{L}$  assigns a differential  $\partial = \partial_1 + \partial_2$  on  $\mathbb{L}(s^{-1}C)$ ; see, e.g., [13, IV.22]. In fact, if  $C$  is a cocommutative differential graded coalgebra viewed as a cocommutative  $A_{\infty}$ -coalgebra, then  $\mathcal{L}_{\infty}(C) = \mathcal{L}(C)$ .

In the rest of the paper  $X$  will always denote a *nilpotent finite CW-complex* and  $Y$  will always denote a *rational finite type CW-complex*, although most of the results can be stated if we remove the finiteness assumption on  $X$ , as in [8, 9].

We recall briefly the Haefliger model [19] of the mapping space via the functorial description of Brown–Szczarba [3]. Let  $B$  be a finite dimensional differential graded algebra model of  $X$  and let  $(\Lambda V, d)$  be a Sullivan (non-necessarily minimal) model of  $Y$ . We denote by  $B^{\sharp}$  the differential coalgebra dual of  $B$  with the grading  $(B^{\sharp})^{-n} = B_n^{\sharp} = \text{Hom}(B^n, \mathbb{Q})$  and consider the free commutative algebra  $\Lambda(\Lambda V \otimes B^{\sharp})$  generated by the  $\mathbb{Z}$ -graded vector space  $\Lambda V \otimes B^{\sharp}$ , endowed with the differential  $d$  induced by the ones on  $(\Lambda V, d)$  and on  $(B^{\sharp}, \delta)$ . Let  $J$  be the differential ideal

generated by  $1 \otimes 1 - 1$ , and the elements of the form

$$v_1 v_2 \otimes \beta - \sum_j (-1)^{|v_2||\beta'_j|} (v_1 \otimes \beta'_j)(v_2 \otimes \beta''_j), \quad v_1, v_2 \in V,$$

where  $\Delta\beta = \sum_j \beta'_j \otimes \beta''_j$ . The inclusion  $V \otimes B^\# \hookrightarrow \Lambda V \otimes B^\#$  induces an isomorphism of graded algebras

$$\rho: \Lambda(V \otimes B^\#) \xrightarrow{\cong} \Lambda(\Lambda V \otimes B^\#)/J,$$

and thus  $\tilde{d} = \rho^{-1}d\rho$  defines a differential in  $\Lambda(V \otimes B^\#)$  and the following holds:

**Theorem 2.3** ([3, 19]). *The commutative differential graded algebra  $(\Lambda(V \otimes B^\#), \tilde{d})$  is a model of  $\text{map}(X, Y)$ , and the commutative differential graded algebra  $(\Lambda(V \otimes B^\#_+, \tilde{d})$  is a model of  $\text{map}^*(X, Y)$ .*

Now write  $B^\# = A \oplus \delta A \oplus H$ , where  $H \cong H(B^\#)$ , with basis  $\{a_j\}$ ,  $\{b_j\}$  and  $\{h_s\}$ . Thus  $\delta a_j = b_j$  and  $\delta h_s = 0$ . Additionally, since  $(\Lambda V, d)$  is a Sullivan algebra, we can choose a basis  $\{v_i\}$  for  $V$  for which  $dv_i \in \Lambda V_{<i}$ . Then we have:

**Lemma 2.4** ([3, 4]). *The commutative differential graded algebra  $(\Lambda(V \otimes B^\#), \tilde{d})$  splits as  $(\Lambda W, \tilde{d}) \otimes \Lambda(U \oplus \tilde{d}U)$ , where*

- (i)  $U$  is generated by  $u_{ij} = v_i \otimes a_j$ ;
- (ii)  $W$  is generated by  $w_{is} = v_i \otimes h_k - x_{is}$ , for suitable  $x_{is} \in \Lambda(V_{<i} \otimes B^\#)$ ;
- (iii)  $\tilde{d}w_{is} \in \Lambda\{w_{ms}\}_{m < i}$ ;
- (iv) if  $\tilde{d}(v_i \otimes h_s)$  is decomposable, so is  $\tilde{d}w_{is}$ .

We can endow the free algebra  $\Lambda(V \otimes H)$  with a differential  $\hat{d}$  so that the map

$$\sigma: (\Lambda(V \otimes H), \hat{d}) \xrightarrow{\cong} (\Lambda W, \tilde{d})$$

is an isomorphism of DGA's, and therefore  $(\Lambda(V \otimes H), \hat{d})$  is a model of  $\text{map}(X, Y)$  called the *reduced Brown-Szczarba model*. The differential  $\hat{d}$  of this model can be easily described (see [4, Lemma 2.8] for the case  $(\Lambda V, d = d_1 + d_2)$ ).

Let  $v \otimes h$  be a generator in  $\Lambda(V \otimes H)$ , and suppose that  $dv = \sum v_1 \cdots v_n$  and  $\Delta^{(n-1)}h = \sum_j c_j^1 \otimes \cdots \otimes c_j^n$ , where  $c_j^i$  are elements of the basis in  $B^\#$ . Consider the sum

$$\tilde{d}(v \otimes h) = \sum_j \sum \epsilon(v_1 \otimes c_j^1) \cdots (v_n \otimes c_j^n),$$

where  $\epsilon$  is the sign given by Koszul convention. If all the elements  $c_j^i$  are in  $H$ , the corresponding term appears in  $\hat{d}(v \otimes h)$  with no modification. If there are some  $c_j^i$  in  $A$ , then the term disappears, and if there are terms in  $\delta A$ , then we must replace the factor  $(v_i \otimes \delta a)$  by the sum

$$\sum_k \sum \epsilon(v_1^i \otimes c_k^1) \cdots (v_{n_i}^i \otimes c_k^{n_i}),$$

where  $dv_i = \sum v_1^i \cdots v_{n_i}^i$  and  $\Delta^{(n_i-1)}a = \sum_k c_k^1 \otimes \cdots \otimes c_k^{n_i}$  and iterate the same process.

If  $C$  is a coalgebra model of  $X$  and  $L$  is an  $L_\infty$ -model of  $Y$ , then we can endow the complex  $\text{Hom}(C, L)$  with an  $L_\infty$ -algebra structure modeling the space of continuous functions from  $X$  to  $Y$ . More concretely,

**Theorem 2.5** ([2, 8]). *The complex of linear maps  $\text{Hom}(C, L)$  with brackets*

$$\begin{aligned}\ell_1(f) &= \ell_1 \circ f + (-1)^{|f|+1} f \circ \delta, \\ \ell_k(f_1, \dots, f_k) &= [-, \dots, -]_L \circ (f_1 \otimes \dots \otimes f_k) \circ \Delta^{(k-1)}, \quad k \geq 2,\end{aligned}$$

*is an  $L_\infty$ -algebra modeling  $\text{map}(X, Y)$ .*

Similarly, the complex  $\text{Hom}(\overline{C}, L)$ , where  $\overline{C} = \ker \varepsilon$  is the kernel of the augmentation  $\varepsilon: C \rightarrow \mathbb{Q}$ , with the same brackets replacing  $\Delta$  with  $\overline{\Delta}$ , is an  $L_\infty$ -model for  $\text{map}^*(X, Y)$  the space of based functions.

### 3. AN EXPLICIT $L_\infty$ -STRUCTURE ON $\text{Hom}(H, L)$

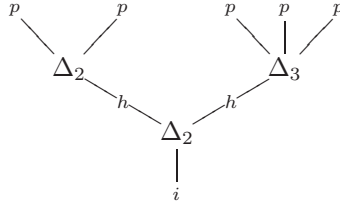
In this section, we describe an explicit  $L_\infty$ -structure on the complex of linear maps  $\text{Hom}(H, L)$ , where  $L$  is an  $L_\infty$ -algebra and  $H$  is the homology of a coalgebra  $C$  with the transferred  $A_\infty$ -structure. We begin by recalling the transference of  $A_\infty$ - and  $L_\infty$ -structures along homotopy retracts.

Let  $(A, d_A)$  and  $(V, d_V)$  be two complexes. We say that  $V$  is a *homotopy retract* of  $A$  if there exist maps

$$\begin{array}{ccc} & & p \\ & \circlearrowleft & \longrightarrow \\ h & (A, d_A) & \xleftarrow{i} (V, d_V) \end{array}$$

such that  $\text{id}_A - ip = d_A h + h d_A$  and  $i$  is a quasi-isomorphism. If  $(A, V, i, p, h)$  is a homotopy retract, then it is possible to transfer  $A_\infty$ - and  $L_\infty$ -structures from  $A$  to  $V$  with explicit formulae. This is in fact a particular instance of the so-called *Homotopy Transfer Theorem* [15, 23, 24, 26, 28], which goes back to [17, 18, 20, 21] for the case of  $A_\infty$ -structures. Before stating the precise statements of these results, we need to introduce some notation on rooted trees.

Let  $T_k$  (resp.  $PT_k$ ) be the set of isomorphism classes of directed rooted trees (resp. planar rooted trees) with internal vertices of valence at least two and exactly  $k$  leaves. Let  $(C, \{\Delta_k\})$  be an  $A_\infty$ -coalgebra and let  $(C, V, i, p, h)$  be a homotopy retract of  $C$ . For each planar tree  $T$  in  $PT_k$ , we define a linear map  $\Delta_T: V \rightarrow V^{\otimes k}$  as follows. The leaves of the tree are labeled by  $p$ , each internal edge is labelled by  $h$  and the root edge is labelled by  $i$ . Every internal vertex  $v$  is labelled by the operation  $\Delta_r$ , where  $r$  is the number of input edges of  $v$ . Moving up from the root to the leaves one defines  $\Delta_T$  as the composition of the different labels. For example, the tree  $T$



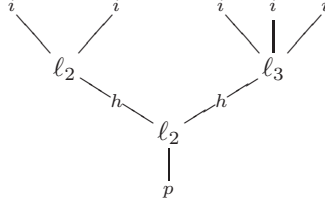
yields the map  $\Delta_T = (((p \otimes p) \circ \Delta_2 \circ h) \otimes ((p \otimes p \otimes p) \circ \Delta_3 \circ h)) \circ \Delta_2 \circ i$ .

Similarly, if  $(L, \{\ell_k\})$  is an  $L_\infty$ -algebra and  $(L, V, i, p, h)$  be a homotopy retract of  $L$ , then each tree  $T$  in  $T_k$  gives rise to a linear map  $\ell_T: \Lambda^k V \rightarrow V$  in the following way. Let  $\tilde{T}$  be a planar embedding of  $T$ . If we label the leaves of the tree by  $i$ , each internal edge by  $h$ , the root edge by  $p$  and each internal vertex by  $\ell_k$ , where  $k$  is the number of input edges, then this planar embedding defines a linear map

$$\ell_{\tilde{T}}: V^{\otimes k} \longrightarrow V$$



by moving down from the leaves to the root, according to the usual operadic rules. For example, for the same tree as before, the labeling reads



and the linear map  $\ell_{\tilde{T}}$  corresponds to

$$p \circ \ell_2 \circ ((h \circ \ell_2 \circ (i \otimes i)) \otimes (h \circ \ell_3 \circ (i \otimes i \otimes i)))$$

Then, we define  $\ell_T$  as the composition of  $\ell_{\tilde{T}}$  with the symmetrization map  $\Lambda^k V \rightarrow V^{\otimes k}$  given by

$$v_1 \wedge \cdots \wedge v_k \mapsto \sum_{\sigma \in S_k} \epsilon_{\sigma} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)},$$

where  $S_k$  denotes the symmetric group on  $k$  letters,  $\epsilon_\sigma$  denotes the signature of the permutation and  $\epsilon$  stands for the sign given by the Koszul convention.

**Theorem 3.1** (Homotopy Transfer). *Let  $(A, d_A)$  and  $(V, d_V)$  be two complexes and let  $(A, V, i, p, h)$  be a homotopy retract of  $A$ . Then the following hold:*

- (i) *If  $A = (C, \{\Delta_k\})$  is an  $A_\infty$ -coalgebra, then there exists an  $A_\infty$ -coalgebra structure  $\{\Delta'_k\}$  on  $V$  and an  $A_\infty$ -quasi-isomorphism*

$$i: (V, \{\Delta'_k\}) \longrightarrow (C, \{\Delta_k\})$$

such that  $\Delta'_1 = d_V$  and  $i^{(1)} = i$ . Moreover, the transferred higher comultiplications can be explicitly described by the formula

$$\Delta'_k = \sum_{T \in PT_k} \Delta_T.$$

- (ii) If  $A = (L, \{\ell_k\})$  is an  $L_\infty$ -algebra, then there exists an  $L_\infty$ -algebra structure  $\{\ell'_k\}$  on  $V$  and an  $L_\infty$ -quasi-isomorphism

$$i: (V, \{\ell'_k\}) \longrightarrow (L, \{\ell_k\})$$

such that  $\ell'_1 = d_V$  and  $i^{(1)} = i$ . Moreover, the transferred higher brackets can be explicitly described by the formula

$$\ell'_k = \sum_{T \in T_k} \frac{\ell_T}{|\text{Aut}(T)|},$$

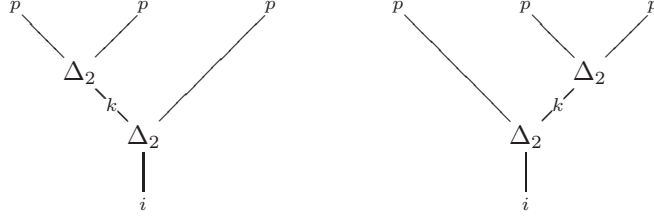
where  $\text{Aut}(T)$  is the automorphism group of the tree  $T$ .

If  $C$  is a cocommutative differential graded coalgebra and  $H \cong H(C)$  denotes the homology of  $C$ , then the transferred  $A_\infty$ -coalgebra structure on  $H$ , whose higher comultiplications are called *higher Massey coproducts* (cf. [26, 10.3.12]) is described as follows. We can decompose  $(C, \delta, \Delta)$  as  $A \oplus \delta A \oplus H$  with basis  $\{a_j\}$ ,  $\{\delta a_j\}$  and  $\{h_s\}$ . Thus,  $\delta = 0$  in  $H$  and  $\delta: A \rightarrow \delta A$  is an isomorphism. This decomposition induces a homotopy retract

$$(3.1) \quad k \circlearrowleft (C, \delta) \xrightleftharpoons[i]{p} (H, 0)$$



given by  $p(a_j) = p(\delta a_j) = 0$ ,  $p(h_s) = h_s$ ;  $i(h_s) = h_s$ ;  $k(a_j) = k(h_s) = 0$  and  $k(\delta a_j) = a_j$ . Then by Theorem 3.1(i), we can transfer the cocommutative differential graded coalgebra structure on  $C$  to an  $A_\infty$ -coalgebra structure on  $H$ . For example, since  $C$  has no higher order coproducts, the operation  $\Delta'_3$  on  $H$  given by the formula of Theorem 3.1(i) is provided by the trees



Explicitly,

$$\begin{aligned}
 \Delta'_3(h) &= (p \otimes p \otimes \text{id}) \circ (\Delta_2 \otimes \text{id}) \circ (k \otimes p) \circ \Delta_2 \circ i(h) \\
 &\quad \pm (\text{id} \otimes p \otimes p) \circ (\text{id} \otimes \Delta_2) \circ (p \otimes k) \circ \Delta_2 \circ i(h) \\
 &= \sum_j (p \otimes p \otimes \text{id}) \circ (\Delta_2 \otimes \text{id}) \circ (k \otimes p)(z'_j \otimes z''_j) \\
 &\quad \pm \sum_j (\text{id} \otimes p \otimes p) \circ (\text{id} \otimes \Delta_2) \circ (p \otimes k)(z'_j \otimes z''_j),
 \end{aligned}$$

where  $\Delta_2(h) = \sum_j z'_j \otimes z''_j$ . For a term of the form  $z'_j \otimes z''_j = \delta a \otimes h'$ , the  $j$ th term in the above summation is

$$\begin{aligned}
 &(p \otimes p \otimes \text{id}) \circ (\Delta_2 \otimes \text{id}) \circ (k(\delta a) \otimes p(h')) \\
 &\quad \pm (\text{id} \otimes p \otimes p)(\text{id} \otimes \Delta_2)(p(\delta a) \otimes k(h')) \\
 &= (p \otimes p \otimes \text{id}) \circ (\Delta_2 \otimes \text{id}) \circ (a \otimes h') \\
 &= \sum_i (p \otimes p \otimes \text{id}) \circ (x'_i \otimes x''_i \otimes h') \\
 &= \sum_i p(x'_i) \otimes p(x''_i) \otimes h',
 \end{aligned}$$

where  $\Delta_2(a) = \sum_i x'_i \otimes x''_i$ .

Replacing  $C$  by the kernel of the augmentation  $\overline{C}$  and using the decomposition  $\overline{C} = A \oplus \delta A \oplus \overline{H}$ , we can proceed similarly as above to obtain a transferred  $A_\infty$ -coalgebra structure on  $\overline{H}$ . Moreover, we have the following:

**Proposition 3.2.** *The transferred  $A_\infty$ -coalgebra structure on  $\overline{H}$  is cocommutative and  $\mathcal{L}_\infty(\overline{H})$  is the Quillen minimal model of  $\mathcal{L}(\overline{C})$ .*

*Proof.* Since the kernel of the augmentation  $\overline{C}$  is cocommutative, the transferred  $A_\infty$ -coalgebra structure on  $\overline{H}$  is also cocommutative (cf. [11, Theorem 12]). Then we can apply the homotopy  $\mathcal{L}_\infty$  Quillen functor to diagram (3.1) and it produces a quasi-isomorphism  $(\mathbb{L}(s^{-1}\overline{H}), \partial) \xrightarrow{\sim} (\mathcal{L}(\overline{C}), \partial)$ . The  $k$ th part  $\partial_k$  of the differential in  $(\mathbb{L}(s^{-1}\overline{H}), \partial)$  is determined by the higher Massey coproduct  $\overline{\Delta}'_k$  in  $\overline{H}$ .

Moreover, by the computations of  $\overline{\Delta}'_k$  made above, if we start by decomposing  $(\overline{C}, \delta, \overline{\Delta})$  as  $A \oplus \delta A \oplus \overline{H}$ , where  $\overline{H} \cong H(\overline{C})$  with basis  $\{a_j\}$ ,  $\{\delta a_j\}$  and  $\{h_s\}$ , then

Figure 1 consists of two Feynman diagrams, labeled  $T_1$  and  $T_2$ , representing tree-level amplitudes. Diagram  $T_1$  shows a process with external momenta  $p^*$ ,  $p^*$ ,  $k^*$ , and  $i^*$ , and internal momenta  $\ell_2$ . Diagram  $T_2$  shows a process with external momenta  $p^*$ ,  $p^*$ ,  $p$ , and  $i^*$ , and internal momenta  $\ell_3$ .

Therefore, if  $f_1$ ,  $f_2$  and  $f_3$  are all elements of  $\text{Hom}(H, L)$  and  $h \in H$ , then  $\ell'_3(f_1, f_2, f_3)(h)$  is expressed in terms of the maps

$$\begin{aligned}\ell_{\tilde{T}_2}(f_1, f_2, f_3)(h) &= i_* \ell_3(p^* f_1, p^* f_2, p^* f_3)(h) \\ &= [-, -, -]_L \circ (p^* f_1 \otimes p^* f_2 \otimes p^* f_3) \circ \Delta^{(2)}(h) \\ &= \sum_j (-1)^{|z'_j|(|f_2|+|f_3|)+|z''_j||f_3|} [f_1 p(z'_j), f_2 p(z''_j), f_3 p(z'''_j)]_L,\end{aligned}$$

where  $\Delta^{(2)}(h) = (\Delta \otimes \text{id}) \circ \Delta(h) = \sum_j z'_j \otimes z''_j \otimes z'''_j$ , and

$$\begin{aligned}\ell_{\tilde{T}_1}(f_1, f_2, f_3)(h) &= i^* \ell_2(k^* \ell_2(p^* f_1, p^* f_2), p^* f_3)(h) \\ &= [-, -]_L \circ (k^* \ell_2(p^* f_1, p^* f_2) \otimes p^* f_3) \circ \Delta(h) \\ &= \sum_j (-1)^{|z'_j||f_3|} [\ell_2(p^* f_1, p^* f_2) \circ k(z'_j), f_3 p(z''_j)]_L,\end{aligned}$$

where  $\Delta(h) = \sum_j z'_j \otimes z''_j$ . For a term of the form  $z'_j \otimes z''_j = \delta a \otimes h'$  the  $j$ th term in the above summation equals

$$\sum_i (-1)^{(|a|+1)|f_3|+|x'_i||f_2|} [[f_1 p(x'_i), f_2 p(x''_i)]_L, f_3 h']_L,$$

where  $\Delta(a) = \sum_i x'_i \otimes x''_i$ .

In order to prove (ii), and in view of the isomorphism  $V \otimes H \cong (s \text{Hom}(H, L))^\sharp$ , it is enough to see that equation (2.1) holds, which in this case amounts to check that

$$\langle \widehat{d}_j(v \otimes h); s f_1 \wedge \cdots \wedge s f_j \rangle = (-1)^\epsilon \langle v \otimes h; s \ell'_j(f_1, \dots, f_j) \rangle.$$

This is a straightforward computation that follows from the explicit  $L_\infty$ -structure on  $\text{Hom}(H, L)$  given in (i) and the description of  $\widehat{d}$  given after Lemma 2.4. For completeness, we make it explicit for the case  $j = 3$ . In what follows we work modulo signs and summations where necessary in order to simplify the computations. By definition

$$\langle v \otimes h; \ell'_3(f_1, f_2, f_3) \rangle = \pm \langle v; \ell_{\tilde{T}_1}(f_1, f_2, f_3)(h) \rangle \pm \langle v; \ell_{\tilde{T}_2}(f_1, f_2, f_3)(h) \rangle.$$

As we have seen before,

$$\begin{aligned}\ell_{\tilde{T}_2}(f_1, f_2, f_3)(h) &= \pm \sum_j [f_1 p(z'_j), f_2 p(z''_j), f_3 p(z'''_j)]_L, \\ \ell_{\tilde{T}_1}(f_1, f_2, f_3)(h) &= \pm [[f_1 p(x'), f_2 p(x'')]_L, f_3 p(h')]_L \pm [[f_1 p(x''), f_2 p(x')]_L, f_3 p(h')]_L,\end{aligned}$$

where  $\Delta(h) = \delta a \otimes h' + h' \otimes \delta a$  and  $\Delta(a) = x' \otimes x'' + x'' \otimes x'$ . Then, applying (2.1) to the  $L_\infty$ -algebra  $L$ , we have on the one hand

$$\begin{aligned}\pm \langle v; \ell_{\tilde{T}_1}(f_1, f_2, f_3)(h) \rangle &= \pm \langle u'; s f_1 p(x') \rangle \langle u''; s f_2 p(x'') \rangle \langle w; s f_3 h' \rangle \\ &\quad \pm \langle u''; s f_1 p(x') \rangle \langle u'; s f_2 p(x'') \rangle \langle w; s f_3 h' \rangle \\ &\quad \pm \langle w'; s f_1 p(x') \rangle \langle w''; s f_2 p(x'') \rangle \langle u; s f_3 h' \rangle \\ &\quad \pm \langle w''; s f_1 p(x') \rangle \langle w'; s f_2 p(x'') \rangle \langle u; s f_3 h' \rangle\end{aligned}$$

$$\begin{aligned}
& \pm \langle u'; sf_1p(x'') \rangle \langle u''; sf_2p(x') \rangle \langle w; sf_3h' \rangle \\
& \pm \langle u''; sf_1p(x'') \rangle \langle u'; sf_2p(x') \rangle \langle w; sf_3h' \rangle \\
& \pm \langle w'; sf_1p(x'') \rangle \langle w''; sf_2p(x') \rangle \langle u; sf_3h' \rangle \\
(3.3) \quad & \pm \langle w''; sf_1p(x'') \rangle \langle w'; sf_2p(x') \rangle \langle u; sf_3h' \rangle,
\end{aligned}$$

where  $d_2v = uw$ ,  $d_2u = u'u''$ , and  $d_2w = w'w''$ , and

$$\begin{aligned}
\pm \langle v; \ell_{\widehat{T}_2}(f_1, f_2, f_3)(h) \rangle &= \pm \sum_j \langle d_3v; sf_1p(z'_j) \wedge sf_2p(z''_j) \wedge sf_3p(z'''_j) \rangle \\
&= \pm \sum_j (\langle q; sf_1p(z'_j) \rangle \langle r; sf_2p(z''_j) \rangle \langle t; sf_3p(z'''_j) \rangle) \\
&\quad \pm \dots\dots\dots \\
(3.4) \quad &\pm \langle t; sf_1p(z'_j) \rangle \langle q; sf_2p(z''_j) \rangle \langle r; sf_3p(z'''_j) \rangle,
\end{aligned}$$

where  $d_3v = qrt$  and the last sum is taken over all possible permutations of the variables  $q$ ,  $r$  and  $t$ . On the other hand, by the definition of  $\widehat{d}$ , we have that only two terms contribute to  $\widehat{d}_3(v \otimes h)$ . Namely, since  $d_3v = qrt$ , the first term is

$$\pm \sum_j (q \otimes p(z'_j))(r \otimes p(z''_j))(t \otimes p(z'''_j)),$$

where  $\Delta^{(2)}(h) = \sum_j z'_j \otimes z''_j \otimes z'''_j$ ; since  $d_2v = uw$ , the second term is

$$\begin{aligned}
& \pm (u' \otimes p(x'))(u'' \otimes p(x''))(w \otimes h') \pm (u' \otimes p(x''))(u'' \otimes p(x'))(w \otimes h') \\
& \pm (u \otimes h')(w' \otimes p(x'))(w'' \otimes p(x'')) \pm (u \otimes h')(w' \otimes p(x''))(w'' \otimes p(x')),
\end{aligned}$$

where  $\Delta(h) = \delta a \otimes h' + h' \otimes \delta a$  and  $\Delta(a) = x' \otimes x'' + x'' \otimes x'$ . The claim follows by observing that

$$\langle \widehat{d}_3(v \otimes h); sf_1 \wedge sf_2 \wedge sf_3 \rangle$$

is precisely (3.3) + (3.4) after applying symmetrization and dividing by the order of the automorphism group of the corresponding tree. The same proof works for the pointed case.  $\square$

#### 4. APPLICATIONS AND EXAMPLES

Rational homotopy of mapping spaces can be studied in terms of invariants of the source and target spaces. In this section, we use homotopy transfer techniques and the  $L_\infty$ -models of mapping spaces described in [2, 8] and Theorem 3.3, in order to improve several known results involving these rational homotopy invariants.

We will work with models of the components of the mapping space, and for a based map  $f: X \rightarrow Y$ , we will denote by  $\text{map}_f^*(X, Y)$  the component containing  $f$ . These  $L_\infty$ -models can be obtained via the process of perturbation and truncation described in Section 2. More explicitly, let  $\varphi: \mathcal{L}(\overline{C}) \rightarrow L$  be an  $L_\infty$ -model of  $f: X \rightarrow Y$ . Then, the composite

$$\psi: \mathcal{L}_\infty(\overline{H}) \longrightarrow \mathcal{L}(\overline{C}) \longrightarrow L$$

is also a model for  $f$ . Hence, as explained in [8], the induced degree  $-1$  linear map  $\overline{C} \rightarrow L$ , which we will keep denoting by  $\varphi$ , is a Maurer–Cartan element of the  $L_\infty$ -algebra  $\text{Hom}(\overline{C}, L)$ , and the induced map  $\overline{H} \rightarrow \overline{C} \rightarrow L$  is a Maurer–Cartan element in  $\text{Hom}(\overline{H}, L)$ . Hence, the perturbed and truncated  $L_\infty$ -algebras  $\text{Hom}(\overline{C}, L)^{(\varphi)}$  and  $\text{Hom}(\overline{H}, L)^{(\psi)}$  are  $L_\infty$ -models for  $\text{map}_f^*(X, Y)$ .

Recall that in what follows,  $X$  will always denote a *nilpotent finite CW-complex* and  $Y$  will always denote a *rational finite type CW-complex*. For a space  $X$ , the *differential length*  $\mathrm{dl}_k(X)$  is the least integer  $n \geq k$  for which the  $n$ th part of the differential of the Sullivan minimal model of  $X$  is non trivial. If there is not such an  $n$ , then  $\mathrm{dl}_k(X) = \infty$ . Note that a space  $X$  is *coformal*, that is, its rational homotopy type is a formal consequence of its homotopy Lie algebra, if and only if  $\mathrm{dl}_3(X) = \infty$ . Indeed,  $X$  is coformal if and only if its minimal Sullivan algebra is quadratic, i.e, is of the form  $(\Lambda V, d_2)$  [29, Proposition 3.3(d)]. Also,  $X$  is (rationally) an  $H$ -space if and only if  $\mathrm{dl}_2(X) = \infty$  [1, Proposition 6.9].

The *bracket length*  $\mathrm{bl}(X)$  is the length of the shortest non-zero iterated bracket in the differential of the Quillen minimal model of  $X$ . If the differential is zero, then  $\mathrm{bl}(X) = \infty$ .

The *rational cone length*  $\mathrm{cl}(X)$  is the least integer  $n$  such that  $X$  has the rational homotopy type of an  $n$ -cone; see [13, p. 359]. The *rational Whitehead length*  $\mathrm{Wl}(X)$  is the length of the longest non-zero iterated Whitehead bracket in  $\pi_{\geq 2}(X) \otimes \mathbb{Q}$ . In particular, if  $\mathrm{Wl}(X) = 1$ , then all Whitehead products vanish.

In [27, Theorem 6.4], Lupton and Smith proved that

$$\mathrm{Wl}(\mathrm{map}_f^*(X, Y)) \leq \mathrm{cl}(X).$$

If  $\mathrm{Wl}(Y) = 1$ , then this bound can be improved [5, Theorem 0.3] by  $\mathrm{cl}(X) - 1$ . If  $Y$  is a coformal space, then [27, Theorem 6.2]:

$$(4.1) \quad \max\{\mathrm{Wl}(\mathrm{map}_f^*(X, Y)), \mathrm{Wl}(\mathrm{map}_f(X, Y))\} \leq \mathrm{Wl}(Y).$$

The Whitehead length of a space can be related with its nilpotency order. Recall that the *rational nilpotency order*  $\mathrm{nil}(X)$  is defined as the nilpotency order  $\mathrm{nil}(L)$  of the minimal  $L_\infty$ -model of  $X$ . For any space  $X$ , we have that  $\mathrm{Wl}(X) \leq \mathrm{nil}(X)$ , and the equality holds if  $X$  is coformal [8, Proposition 4.2]. Thus, inequality (4.1) can be generalized to [8, Corollary 1.5]:

$$\max\{\mathrm{Wl}(\mathrm{map}_f^*(X, Y)), \mathrm{Wl}(\mathrm{map}_f(X, Y))\} \leq \mathrm{nil}(Y).$$

Here we show that we can replace  $\mathrm{Wl}$  by  $\mathrm{nil}$  in the above inequality, thus improving [8, Corollary 4.3], where  $X$  was assumed to be formal and  $f$  the constant map.

**Theorem 4.1.** *Let  $X$  be a nilpotent finite CW-complex and let  $Y$  be a rational finite type CW-complex. For any  $f: X \rightarrow Y$  we have the inequality*

$$\max\{\mathrm{nil}(\mathrm{map}_f^*(X, Y)), \mathrm{nil}(\mathrm{map}_f(X, Y))\} \leq \mathrm{nil}(Y).$$

The proof relies on the following lemma.

**Lemma 4.2.** *Let  $L$  be an  $L_\infty$ -algebra. The following holds:*

- (i) *If  $z \in L_{-1}$  is a Maurer–Cartan element, then  $\mathrm{nil}(L^{(z)}) \leq \mathrm{nil}(L)$ .*
- (ii) *If  $L'$  is a homotopy retract of  $L$  endowed with the transferred  $L_\infty$ -algebra structure described in Theorem 3.1, then  $\mathrm{nil}(L') \leq \mathrm{nil}(L)$ .*
- (iii) *If  $C$  is a differential graded cocommutative coalgebra, then*

$$\mathrm{nil}(\mathrm{Hom}(C, L)) \leq \mathrm{nil}(L).$$

*Proof.* The proof is straightforward by comparing the explicit definition of the higher brackets in each case. To prove part (i), use that each  $\ell_k^z$  is a perturbation of  $\ell_k$ , that is,  $\ell_k^z(x_1, \dots, x_k) = \ell_k(x_1, \dots, x_k) + \Phi$ , where  $\Phi \in F^{>k}L$ . Part (ii) is proved

by observing that  $F^k L' = 0$  if  $F^k L = 0$  as a consequence of the explicit formula for  $\ell'$  given in Theorem 3.1(ii). Finally, part (iii) follows from [8, Proposition 4.2].  $\square$

*Proof of Theorem 4.1.* Let  $C$  be a coalgebra model for  $X$  and let  $L$  be the minimal  $L_\infty$ -model of  $Y$ . Let  $\phi: C \rightarrow L$  be a Maurer–Cartan element representing the map  $f$ . Then,  $\text{Hom}(C, L)^{(\phi)}$  is an  $L_\infty$ -algebra model for  $\text{map}_f(X, Y)$ , by [2, 8]. We can find a homotopy retract between  $\text{Hom}(C, L)^{(\phi)}$  and its homology  $H_*(\text{Hom}(C, L)^{(\phi)})$ , and applying Theorem 3.1 we obtain a minimal  $L_\infty$ -structure  $\{\ell'_k\}$  on  $H_*(\text{Hom}(C, L)^{(\phi)})$ , exhibiting it as the minimal  $L_\infty$ -model for  $\text{map}_f(X, Y)$ . Now, applying Lemma 4.2, we see that

$$\begin{aligned} \text{nil}(\text{map}_f(X, Y)) &= \text{nil}(H_*(\text{Hom}(C, L)^{(\phi)})) \leq \text{nil}(\text{Hom}(C, L)^{(\phi)}) \\ &\leq \text{nil}(\text{Hom}(C, L)) \leq \text{nil}(L) = \text{nil}(Y). \end{aligned}$$

The same proof works for  $\text{map}_f^*(X, Y)$  after replacing  $C$  with  $\overline{C}$ .  $\square$

Another interesting question is to determine whether a mapping space is of the rational homotopy type of an  $H$ -space, i.e., it has a Sullivan minimal model with zero differential, in terms of the source and target spaces.

In [14, Theorem 2] a necessary condition, based on the Toomer invariant, is given in order to ensure that the component of the constant map is an  $H$ -space. In [6, Theorem 4] it was proved that if  $\text{cl}(X) < \text{dl}_2(Y)$  then *all* the components  $\text{map}_f^*(X, Y)$  are rationally  $H$ -spaces.

If all Whitehead brackets in  $\pi_*(Y)$  are zero, then  $\text{dl}_2(Y) > 2$  and hence  $\text{dl}_2(Y) = \text{dl}_3(Y)$ . If  $\text{dl}_2(Y) = 2$ , then  $\text{dl}_2(Y) < \text{dl}_3(Y)$  and the following result improves [6, Theorem 4].

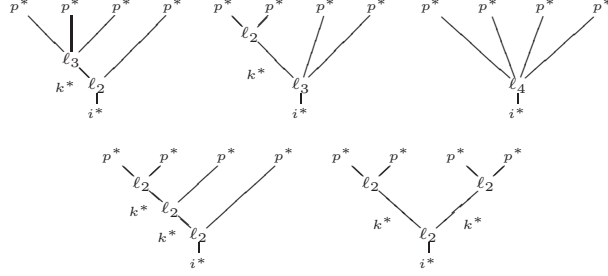
**Theorem 4.3.** *If  $\text{cl}(X) < \text{dl}_3(Y)$  and  $\text{Wl}(Y) < \text{bl}(X)$ , then all the components of the mapping space  $\text{map}^*(X, Y)$  are rationally  $H$ -spaces.*

*Proof.* Let  $(L, \{\ell_k\})$  be the minimal  $L_\infty$ -model of  $Y$ . Thus,  $\ell_k = 0$  for all  $2 < k < \text{dl}_3(Y)$ . Following [12], we may choose  $C$  a coalgebra model of  $X$  such that the conilpotence of  $\overline{C}$  is  $\text{cl}(X)$ , i.e., such that the iterated coproducts of length greater or equal than  $\text{cl}(X)$  are zero. Consider the  $L_\infty$ -model  $\text{Hom}(\overline{H}, L)^{(\psi)}$  of the component corresponding to  $\psi$  given in Theorem 3.3. In order to prove that  $\text{map}_f^*(X, Y)$  is an  $H$ -space we will show that all brackets  $\ell'_k{}^\psi$  for  $k \geq 2$  vanish, and thus  $\mathcal{C}^\infty(\text{Hom}(\overline{H}, L)^{(\psi)})$  is of the form  $(\Lambda S, d_1)$ , for which  $(\Lambda H(S, d_1), 0)$  is a Sullivan minimal model. It is enough to check that  $\ell'_k{}^\psi = 0$  for all  $k$  since this implies that  $\ell'_k{}^\psi = 0$  for all  $k$ .

Now, all the brackets are represented by summations on trees. But due to the relation between the conilpotence condition on  $\overline{C}$  and  $\text{dl}_3(Y)$  one can check that non binary trees do not contribute to the brackets.

By Proposition 3.2 any binary tree, when applied to an element  $h \in \overline{H}$ , yields an iterated Whitehead bracket in  $L$  of the length of some term in  $\partial s^{-1}h$ , where  $\partial$  is the differential of the minimal model of  $X$ . By the condition  $\text{Wl}(Y) < \text{bl}(X)$  the above term also vanishes.  $\square$

We illustrate the last part of the proof by showing that  $\ell'_4{}^\psi = 0$ . This operation is defined as a sum of maps whose terms are indexed by the following five trees:



Note that the first three trees do not contribute to  $\ell'_4$  since  $\text{cl}(X) < \text{dl}_3(Y)$ . For example, for the second tree we have that

$$\begin{aligned} & i^* \ell_3(k^* \ell_2(p^* f_1, p^* f_2), p^* f_3, p^* f_4)(h) \\ &= [-, -, -]_L \circ (k^* \ell_2(p^* f_1, p^* f_2) \otimes p^* f_3 \otimes p^* f_4) \circ \overline{\Delta}^{(2)}(h) \end{aligned}$$

is zero. Indeed, if  $\overline{\Delta}^{(2)}(h) \neq 0$  then  $2 < \text{cl}(X) < \text{dl}_3(Y)$ . This forces  $\text{dl}_3(Y) \geq 4$  and hence  $[-, -, -]_L = 0$ .

The last two trees do not contribute to  $\ell'_4$  either, since  $\text{Wl}(Y) < \text{bl}(X)$ . Thus, for example, for any  $h \in \overline{H}$ , the fourth tree yields

$$i^* \ell_2(k^* \ell_2(k^* \ell_2(p^* f_1, p^* f_2), p^* f_3), p^* f_4)(h).$$

A generic term in this expression is given by

$$(4.2) \quad \sum_i [[f_1 p x'_i, f_2 p x''_i]_L, f_3 p y''_j]_L, f_4 p z''_l]_L,$$

where  $\overline{\Delta}(h) = \sum_l z'_l \otimes z''_l$ ,  $z'_l = \delta a$  and hence  $k z'_l = a$ ,  $\overline{\Delta}(a) = \sum_j y'_j \otimes y''_j$ ,  $y'_j = \delta a'$  and hence  $k y'_j = a'$ , and  $\overline{\Delta}(a') = \sum_i x'_i \otimes x''_i$ . If  $p x'_i, p x''_i, p y''_j, p z''_l \neq 0$ , then by the recursive formula (3.2) of Proposition 3.2, the term

$$[[s^{-1} p x'_i, s^{-1} p x''_i], s^{-1} p y''_j], s^{-1} p z''_l]$$

appears in the expression of  $\partial s^{-1} h$ , where  $\partial$  is the differential of the minimal Quillen model of  $X$ . Therefore, (4.2) must be zero, since  $\text{Wl}(Y) < \text{bl}(X)$ .

*Remark 4.4.* We can give an alternative proof of Theorem 4.3 by relying on [4, Theorem 1.4(2)], where the same result is obtained, but under the assumptions  $\text{Wl}(Y) < \text{bl}(X)$  and coformality of the target space  $Y$ . Indeed, let  $L$  be a minimal  $L_\infty$ -model of  $Y$  and denote by  $L^{\text{cof}}$  the differential graded Lie algebra obtained from  $L$  by discarding all higher brackets except the binary one. The graded Lie algebra  $L^{\text{cof}}$  represents a coformal rational space  $Y^{\text{cof}}$  such that  $\text{Wl}(Y) = \text{Wl}(Y^{\text{cof}})$ . Let  $C$  be a coalgebra model of  $X$  such that  $\overline{C}$  has conilpotency  $\text{cl}(X)$ , then  $\text{cl}(X) < \text{dl}_3(Y)$  implies that the convolution  $L_\infty$ -algebra  $\text{Hom}(\overline{C}, L)$  is isomorphic to the convolution differential graded Lie algebra  $\text{Hom}(\overline{C}, L^{\text{cof}})$ . Therefore  $\text{map}^*(X, Y) \simeq \text{map}^*(X, Y^{\text{cof}})$ .

*Example 4.5.* Let  $X = S_a^3 \vee S_b^3 \cup_\gamma e^8$ , let  $\alpha, \beta \in \pi_3(X)$  be the elements represented by  $S_a^3$  and  $S_b^3$ , respectively, and let  $\gamma = [\alpha, [\alpha, \beta]] \in \pi_7(X)$ . Then, the minimal Quillen model for  $X$  is  $(\mathbb{L}(W), \partial)$ , where  $W = \langle a, b, c \rangle$ ,  $|a| = |b| = 2$  and the only non-zero differential is  $\partial c = [a, [a, b]]$ . Hence, it is clear that  $\text{bl}(X) = 3$ . Moreover,  $\text{cl}(X) = 2$  since  $W = W_0 \oplus W_1$  with  $\partial W_0 = 0$  and  $\partial W_1 \subseteq \mathbb{L}(W_0)$ ; see, for example, [13, Theorem 29.1].



Let  $k_3: Y_2 = \mathbb{C}P_{\mathbb{Q}}^{\infty} \rightarrow K(\mathbb{Q}, 4)$  be the map whose model is given by the map  $(\Lambda x; 0) \leftarrow (\Lambda v; 0)$  that sends  $v$  to  $x^2$ , where  $|x| = 2$ . Let  $k_5: Y_3 = Y_4 = S_{\mathbb{Q}}^2 \rightarrow K(\mathbb{Q}, 6)$  be the map whose model is given by the map  $(\Lambda x, y; d) \leftarrow (\Lambda w; 0)$  that sends  $w$  to  $x^3$ , where  $|y| = 3$ . Now let  $Y = Y_5$  be the rational Postnikov piece

$$Y = Y_5 \xrightarrow{p_5} Y_4 = Y_3 \xrightarrow{p_3} Y_2$$

defined by the  $k$ -invariants  $k_3$  and  $k_5$  and thus modeled by

$$(\Lambda x; 0) \leftarrow (\Lambda x, y; d) \leftarrow (\Lambda x, y, z; d),$$

where  $dy = x^2$  and  $dz = x^3$ . Therefore,  $Y$  is a non-coformal space with  $\mathrm{dl}_2(Y) = \mathrm{Wl}(Y) = 2$  and  $\mathrm{dl}_3(Y) = 3$ . Moreover, there are non null-homotopic maps from  $X$  to  $Y$ . For example  $\phi: \Lambda V \rightarrow \mathcal{C}^{\infty}(\mathbb{L}(a, b, c)) = \Lambda s\mathbb{L}(a, b, c)^{\sharp}$ , defined by  $\phi(x) = \phi(z) = 0$  and  $\phi(y) = sa^{\sharp}$ .

Then, by Theorem 4.3, *every* component of the mapping space  $\mathrm{map}^*(X, Y)$  is rationally an  $H$ -space. This example provides a situation where we cannot apply neither [6, Theorem 4], since  $\mathrm{cl}(X) = \mathrm{dl}_2(Y)$ , nor [4, Theorem 1.4(2)], since  $Y$  is not coformal. Also [14, Theorem 2] does not provide any information for the component of a non-constant map.

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